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Relation Between Quasirigidity and *L*-Rigidity in Space-Times of Constant Curvature and Weak Fields

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The relation between quasirigidity and *L*-rigidity in space-times of constant nonzero curvature and in space-times with small curvature (weak fields) is studied. The covariant expansion of bitensors about a point is considered. We obtain an increase in the order of magnitude, under *L*-rigidity conditions, of the rate of change with respect to a comoving orthonormal frame of the linear momentum, angular momentum, and reduced multipole moments of the energy-momentum tensor. Thus, *L*-rigidity leads to quasirigidity in such space-times.

1. INTRODUCTION

In a preceding paper (Barreda and Olivert, 1996), a definition of *L*-rigidity was proposed as a special class of pseudorigid motions (Ehlers and Rudolph, 1977; Köhler and Schattner, 1979) and therefore it depended on the chosen curve *L*. In *L*-rigid motion, the expansion of the vector field that described the pseudorigid motion vanished. Another kinematical condition required that the Lie derivative of the normal 1-form to the pseudorigid "body" world-tube, with respect to the vector field that described the pseudorigid motion, vanished. The third condition was a dynamical one: The family of tensor fields along *L* obtained by parallely transporting the energy-momentum tensor from the world lines that constituted the pseudorigid motion to the "center of motion" *L* had constant components with respect to a comoving orthonormal frame.

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In that paper (Barreda and Olivert, 1996), we studied the conditions of L-rigidity by applying to them some techniques of the PPN formalism in general relativity. It was shown that the proposed idealization was approximately satisfied by a wide range of real physical systems. The obtained results agreed with classical rigidity, in the sense that the baryon mass density was constant in first order and the stress tensor was constant in the comoving system. Moreover, the Newtonian potential was constant along the line L and the gravitational field was constant along the line L in the comoving system. By choosing L as the center-of-mass line in the Minkowski space-time, we obtained that L-rigidity and weak rigidity (Del Olmo and Olivert, 1983, 1985, 1986, 1987) were equivalent when the angular velocity was both small and constant. In this direction, a new step is to study the relation between quasirigidity and L-rigidity.

In the present work our aim is to study the conditions of L-rigidity in space-times of constant curvature and in space-times with small curvature by applying to them some techniques of the covariant expansion of bitensors about a point (DeWitt and Brehme, 1960). We will check the way in which they modify the rate of change with respect to a comoving orthonormal frame of the reduced multipole moments of the energy-momentum tensor.

In Section 2 we give some remarks on our notation and general assumptions.

In Section 3 we begin by recalling L-rigidity equations and their expressions in a bitensorial form. Subsequently, we study the evolution of the multipole moments of the energy-momentum tensor defined from the parallel propagator (Dixon, 1964). We show in Theorem 1 that the moments $\mathsf{t}^{\cdots}$, given in equation (14), have constant components with respect to a comoving orthonormal frame under L-rigidity conditions. Theorem 2 shows that the moments $\mathsf{p}^{\cdots}$, given in equation (13), also have constant components with respect to a comoving orthonormal frame under L-rigidity conditions. In this last theorem, we suppose that the vector field family $\bar{N}^{\kappa}$ [equation (18)] along L, obtained by parallely transporting the normal field to the pseudorigid "body" world-tube, from the world lines that constitute the pseudorigid motion to the "center of motion" L, has constant components with respect to a comoving orthonormal frame. We show that this supposition is not a restriction when we particularize to space-times of constant curvature and to space-times with small curvature. Thus, L-rigidity, under the last condition, leads to the result that the multipole moments of the energy-momentum tensor defined from the parallel propagator have constant components with respect to a comoving orthonormal frame. In this sense quasirigidity and L-rigidity are analogous concepts since quasirigidity imposes that the reduced multipole moments of the energy-momentum tensor have constant components with respect to a comoving orthonormal frame.

In Section 4 we study the relation between quasirigidity and L-rigidity for the following two cases: (1) space-times of constant curvature, (2) space-times with small curvature (weak fields).

We begin with space-times of constant curvature. From the Riemann tensor expression we get the covariant expansion of bitensors that take part in the definition of the reduced multipole moments of the energy-momentum tensor. We obtain that these moments are expressed from the multipole moments defined from the parallel propagator. On the other hand, we get that the vector field family $\bar{N}^\kappa$ along L satisfies rotating M-transport equations to $O(\mathsf{s}^2)$ and so the moments $\mathsf{p}^{\cdots}$ have constant components, to this approximation order, under L-rigidity conditions.

Likewise the linear momentum and the angular momentum are expressed from the moments $\mathsf{p}^{\cdots}$, and so the linear momentum and the angular momentum have constant components, under L-rigidity conditions, with respect to a comoving orthonormal frame to $O(\mathsf{s}^2)$ and $O(\mathsf{s}^3)$, respectively.

On the other hand, by considering the base line L as the center-of-mass line we obtain that L-rigidity leads to quasirigidity to $O(\mathsf{s}^3)$.

Then we study L-rigidity in weak fields. Using the same techniques as in the case of space-times of constant curvature, we also express the reduced multipole moments of the energy-momentum tensor from the moments $\mathsf{t}^{\cdots}$ and $\mathsf{p}^{\cdots}$. Moreover, the vector field family $\bar{N}^\kappa$ along L satisfies rotating M-transport equations to $O(\mathsf{s})$, and thus the moments $\mathsf{p}^{\cdots}$ have constant components, to $O(\mathsf{s}^2)$, under L-rigidity conditions.

In addition, we get that the linear momentum and the angular momentum are expressed from the moments $\mathsf{p}^{\cdots}$, and thus they have constant components with respect to a comoving orthonormal frame, under L-rigidity conditions, to $O(\mathsf{s})$ and $O(\mathsf{s}^2)$, respectively. As a consequence, we get that the mass is constant, under L-rigidity conditions, to $O(\mathsf{s})$.

Moreover, by considering the base line L as the center-of-mass line we obtain that L-rigidity leads to quasirigidity to $O(\mathsf{s}^2)$.

2. NOTATION

The notation used in this paper is basically that of Barreda and Olivert (1996). Greek indices range from 1 to 4.

We denote partial differentiation by $(\cdot)_{,\alpha}$ and covariant differentiation by $(\cdot)_{;\alpha}$ or ∇_α. Symmetrization and antisymmetrization of indices are denoted by $(\cdot)$ and $[\cdot]$, respectively, while indices enclosed between vertical lines are omitted from these operations.

The Riemann curvature tensor is defined by

$$X^\alpha{}_{;[\beta\gamma]} = -\tfrac{1}{2} X^\delta R^\alpha{}_{\delta\beta\gamma}$$

We will use the theory of bitensors developed by Synge (1966) and DeWitt and Brehme (1960). For a bitensor function of the point pair (z, m) we distinguish between indices at z and indices at m. We will use $\kappa, \lambda, \ldots$ as indices at z and $\alpha, \beta, \ldots$ at m.

We denote by $\sigma(z, m)$ the world function and by

$$H^{\alpha}{}_{\lambda} = (-\sigma_{.}{}^{\lambda}{}_{\alpha})^{-1}, \qquad K^{\alpha}{}_{\lambda} = H^{\alpha}{}_{\mu}\sigma_{.}{}^{\mu}{}_{\lambda}$$

the Jacobi propagators (Dixon, 1970). In this notation, a dot followed by an index denotes covariant differentiation with respect to one or another variable.

Another important bitensor that we will use is the parallel propagator, denoted by $\bar{g}_{\lambda\alpha}(z, m)$.

We want to give some covariant expansions of bitensors. The second covariant derivative of the world function with respect to the variables z and m and the Jacobi propagators have the following expansions:

$$\sigma_{.\lambda\alpha} = -\bar{g}_{\lambda\alpha} - \tfrac{1}{6}\bar{g}_{\lambda}{}^{\beta}R_{\beta\gamma\alpha\delta}\sigma_{.}{}^{\gamma}\sigma_{.}{}^{\delta} + O(\mathsf{s}^3)$$

$$H_{\alpha\lambda} = \bar{g}_{\lambda\alpha} - \tfrac{1}{6}\bar{g}_{\lambda}{}^{\beta}R_{\beta\gamma\alpha\delta}\sigma_{.}{}^{\gamma}\sigma_{.}{}^{\delta} + O(\mathsf{s}^3)$$

$$K_{\alpha\lambda} = \bar{g}_{\lambda\alpha} - \tfrac{1}{2}\bar{g}_{\lambda}{}^{\beta}R_{\alpha\gamma\beta\delta}\sigma_{.}{}^{\gamma}\sigma_{.}{}^{\delta} + O(\mathsf{s}^3)$$

where $\mathsf{s}(z, m)$ is the biscalar of geodetic interval, which gives the magnitude of the invariant distance between z and m as measured along a geodesic joining them.

We also recall that with each pair (L, n), where L is a timelike curve parametrized as $z(s)$ and n is a timelike unit vector field along L, there is associated a collection of hypersurfaces $\Sigma(s)$ formed by geodesics through $z(s)$ orthogonal to $n(s)$. Since these hypersurfaces are disjoint, there is a well-defined differentiable function (Dixon, 1974) $\chi(m)$ on $\Sigma = \cup_s \Sigma(s)$ given by

$$\chi(m) = s \qquad \text{if} \quad m \in \Sigma(s)$$

This function allows us to connect bitensors with ordinary tensor fields on $\mathcal{M}$, where $\mathcal{M}$ is the space-time manifold. In order to simplify the notation, we consider

$$s_m = \chi(m), \qquad z_m = z(\chi(m))$$

Dixon (1974) defines, relative to the pair (L, n), the linear momentum p^{κ}, the angular momentum $S^{\kappa\lambda}$, and the reduced multipole moments of the energy-momentum tensor $J^{\kappa_1\cdots\kappa_n\lambda\mu\nu\rho}$ as integrals on hypersurfaces $\Sigma(s)$. In Dixon's notation

$$J^{\kappa_1\cdots\kappa_n\lambda\mu\nu\rho} = t^{\kappa_1\cdots\kappa_n[\lambda[\nu\mu]\rho]} + \frac{1}{n+1}p^{\kappa_1\cdots\kappa_n[\lambda[\nu\mu]\rho]\tau}\dot{z}_{\tau}, \qquad n \geq 0$$

where the nested square brackets denote here antisymmetrization on λ, μ and ν, ρ independently. The time variation of the $J^{\cdots}$ remains unspecified.

Using these definitions, Schattner (1978a,b) proves, under a supposition of weak field, the existence and uniqueness of a pair (L_o, n) such that

$$0 = S^{\lambda\mu}p_\mu, \qquad p^\lambda = Mn^\lambda$$

The curve L_o is called the center-of-mass line.

In order to obtain simple models of bodies that remove the arbitrariness of the $J^{\cdots}$ evolution, Dixon (1970) and Ehlers and Rudolph (1977) define dynamical rigidity or quasirigidity: A motion is quasirigid if the moments $J^{\cdots}$ have constant components with respect to a comoving orthonormal frame. In our notation

$$\overset{\text{RM}}{D}_n J^{\cdots} = 0$$

where

$$\overset{\text{RM}}{D}_n X^\lambda = \dot{X}^\lambda + M^\lambda{}_\mu X^\mu, \qquad M^\lambda{}_\mu = \dot{n}^\lambda n_\mu - n^\lambda \dot{n}_\mu + \Omega^\lambda{}_\mu$$

is the derivative operator associated with rotating M-transport, and $\Omega_{\lambda\mu}$ is a skew tensor along L that describes the angular velocity of the comoving frame relative to an M-transported frame. A comoving orthonormal frame is an orthonormal tetrad, having n^κ as one vector, that satisfies rotating M-transport.

3. MULTIPOLE MOMENTS AND L-RIGIDITY

The equations which define the L-rigidity, proposed by Barreda and Olivert (1996), are given by

$$\nabla \cdot \omega = 0 \tag{1}$$

$$(\overset{\text{RM}}{D}_n \bar{T})(s_m, m) = 0 \qquad \forall m \in \Sigma \tag{2}$$

$$\iota_N \iota_N \mathfrak{L}_\omega g = 0 \tag{3}$$

where

$$\omega^\alpha = K^\alpha{}_\kappa \dot{z}^\kappa + H^\alpha{}_\kappa M^\kappa{}_\lambda \sigma^\lambda \tag{4}$$

is the vector field that describes the pseudorigid motion, and

$$\bar{T}^{\lambda\mu}(s, m) = \bar{g}^\lambda{}_\alpha(z(s), \gamma_m(s - s_m))\bar{g}^\mu{}_\beta(z(s), \gamma_m(s - s_m))T^{\alpha\beta}(\gamma_m(s - s_m)) \tag{5}$$

is the tensor field family along L obtained by parallely transporting the energy-momentum tensor from the integral curves γ_m ($m \in \Sigma$) of ω^α to the "center of motion" L. Moreover, N^{α} is a vector field on the world-tube Σ that coincides with the unit normal vector field on each hypersurface $\Sigma(s)$.

In Barreda and Olivert (1996) we obtained the L-rigidity equations in a bitensorial form, given by

$$K^{\alpha}{}_{\kappa.\alpha}\dot{z}^{\kappa} + H^{\alpha}{}_{\kappa.\alpha}M^{\kappa}{}_{\lambda}\sigma_{.}{}^{\lambda} + (K^{\alpha}{}_{\kappa.\lambda}\dot{z}^{\lambda}\dot{z}^{\kappa} + K^{\alpha}{}_{\kappa}\ddot{z}^{\kappa} + H^{\alpha}{}_{\kappa.\lambda}\dot{z}^{\lambda}M^{\kappa}{}_{\mu}\sigma_{.}{}^{\mu} + H^{\alpha}{}_{\kappa}\dot{M}^{\kappa}{}_{\lambda}\sigma_{.}{}^{\lambda} + H^{\alpha}{}_{\kappa}M^{\kappa}{}_{\lambda}\sigma_{.}{}^{\lambda}{}_{\mu}\dot{z}^{\mu})\chi_{,\alpha} = 0 \quad (6)$$

$$2\dot{z}^{\nu}\bar{g}^{(\lambda}{}_{\alpha.\nu}\bar{g}^{\mu)}{}_{\beta}T^{\alpha\beta} + 2\omega^{\gamma}\bar{g}^{(\lambda}{}_{\alpha.\gamma}\bar{g}^{\mu)}{}_{\beta}T^{\alpha\beta} + \bar{g}^{\lambda}{}_{\alpha}\bar{g}^{\mu}{}_{\beta}T^{\alpha\beta}{}_{;\gamma}\omega^{\gamma} + 2\bar{g}^{(\lambda}{}_{\alpha}M^{\mu)}{}_{\rho}\bar{g}^{\rho}{}_{\beta}T^{\alpha\beta} = 0 \quad (7)$$

$$n_{\nu}\sigma_{.}{}^{\nu\beta}\{n^{\rho}\sigma_{.\rho\alpha}[K^{\alpha}{}_{\kappa.\beta}\dot{z}^{\kappa} + H^{\alpha}{}_{\kappa.\beta}M^{\kappa}{}_{\lambda}\sigma_{.}{}^{\lambda} - \dot{n}_{\lambda}\sigma_{.}{}^{\lambda}{}_{\beta} + (K^{\alpha}{}_{\kappa.\lambda}\dot{z}^{\lambda}\dot{z}^{\kappa} + H^{\alpha}{}_{\kappa.\lambda}\dot{z}^{\lambda}M^{\kappa}{}_{\mu}\sigma_{.}{}^{\mu}),\chi_{,\beta}] - (n_{\kappa}\dot{M}^{\kappa}{}_{\lambda}\sigma_{.}{}^{\lambda} + n_{\lambda}\sigma_{.}{}^{\lambda}{}_{\mu}\ddot{z}^{\mu} + \dot{n}_{\lambda}\sigma_{.}{}^{\lambda}{}_{\mu}\dot{z}^{\mu})\chi_{,\beta}\} = 0 \quad (8)$$

On the other hand, the linear and angular momenta were first given by Dixon (1964) from the bitensor $\bar{g}^{\lambda}{}_{\alpha}$ instead the bitensors $H^{\alpha}{}_{\lambda}$ and $K^{\alpha}{}_{\lambda}$. In a subsequent paper Dixon (1970) indicated that these definitions were made without sufficient study of their implications. Even so, we are interested in considering these moments with the purpose of studying the analogy between quasirigidity and L-rigidity. These moments are defined by

$$\mathrm{p}^{\lambda}(s) = \int_{\Sigma(s)} \bar{g}^{\lambda}{}_{\alpha}T^{\alpha\beta}\, d\Sigma_{\beta} \quad (9)$$

$$\mathrm{S}^{\lambda\mu}(s) = 2\int_{\Sigma(s)} \sigma_{.}{}^{[\mu}\bar{g}^{\lambda]}{}_{\alpha}T^{\alpha\beta}\, d\Sigma_{\beta} \quad (10)$$

and satisfy the equations

$$\dot{\mathrm{p}}^{\lambda} = \tfrac{1}{2}R^{\lambda}{}_{\nu\mu\rho}(\mathrm{t}^{\rho\nu\mu} + \dot{z}^{\mu}\mathrm{p}^{\rho\nu}) + O(\mathrm{s}^2) \quad (11)$$

$$\dot{\mathrm{S}}^{\lambda\mu} = 2\mathrm{p}^{[\lambda}\dot{z}^{\mu]} + R^{[\lambda}{}_{\rho\nu\tau}(\tfrac{1}{3}\mathrm{t}^{|\rho\tau|\mu]\nu} - \mathrm{t}^{\mu]\tau\rho\nu} + \tfrac{2}{3}\mathrm{p}^{|\rho\tau|\mu]}\dot{z}^{\nu} - \mathrm{p}^{\mu]\tau\rho}\dot{z}^{\nu}) + O(\mathrm{s}^3) \quad (12)$$

where

$$\mathrm{p}^{\kappa_1\cdots\kappa_n\lambda}(s) = (-1)^n\int_{\Sigma(s)} \sigma_{.}{}^{\kappa_1}\cdots\sigma_{.}{}^{\kappa_n}\bar{g}^{\lambda}{}_{\alpha}T^{\alpha\beta}\, d\Sigma_{\beta}, \qquad n \geq 1 \quad (13)$$

$$\mathrm{t}^{\kappa_1\cdots\kappa_n\lambda\mu}(s) = (-1)^n\int_{\Sigma(s)} \sigma_{.}{}^{\kappa_1}\cdots\sigma_{.}{}^{\kappa_n}\bar{g}^{\lambda}{}_{\alpha}\bar{g}^{\mu}{}_{\beta}T^{\alpha\beta}\omega^{\gamma}\, d\Sigma_{\gamma}, \qquad n \geq 1 \quad (14)$$

are the multipole moments of the energy momentum-tensor defined from the parallel propagator. Obviously, equation (13) for $n = 0$ coincides with the expression given in (9).

In this way, the moments $\mathfrak{p}^{\cdots}$ and $\mathfrak{t}^{\cdots}$ play the same role as the reduced multipole moments of the energy-momentum tensor $J^{\cdots}$, given that they also have an arbitrary time variation. We will show that this arbitrariness is removed in the *L*-rigidity concept.

We begin with the evolution of the moments $\mathfrak{t}^{\cdots}$.

Theorem 1. The moments $\mathfrak{t}^{\cdots}$ have constant components with respect to a comoving orthonormal frame in *L*-rigid motion.

Proof. As the flow φ_s of the vector field ω^α (Barreda and Olivert, 1996) is a diffeomorphism that drags $\Sigma(s_o)$ into $\Sigma(s_o + s)$ and it preserves the orientation, for sufficiently small s and s_o arbitrary, equation (1) leads to

$$\mathfrak{t}^{\kappa_1\cdots\kappa_n\lambda\mu}(s_o + s) = (-1)^n \int_{\Sigma(s_o)} (\sigma_{\cdot}^{\kappa_1} \cdots \sigma_{\cdot}^{\kappa_n}\bar{g}^{\lambda}{}_{\alpha}\bar{g}^{\mu}{}_{\beta}T^{\alpha\beta}) \circ \varphi_s \iota_\omega \eta \tag{15}$$

Taking the derivative of the last equation at $s = 0$, we get

$$\begin{aligned}
\left.\frac{d\mathfrak{t}^{\kappa_1\cdots\kappa_n\lambda\mu}}{ds}\right|_{s_o} &= n(-1)^n \int_{\Sigma(s_o)} \dot{z}^{\nu}\sigma_{\cdot}^{(\kappa_1}{}_{,\nu}\sigma_{\cdot}^{\kappa_2} \cdots \sigma_{\cdot}^{\kappa_n)}\bar{g}^{\lambda}{}_{\alpha}\bar{g}^{\mu}{}_{\beta}T^{\alpha\beta}\iota_\omega\eta \\
&+ n(-1)^n \int_{\Sigma(s_o)} \omega^{\gamma}\sigma_{\cdot}^{(\kappa_1}{}_{,\gamma}\sigma_{\cdot}^{\kappa_2} \cdots \sigma_{\cdot}^{\kappa_n)}\bar{g}^{\lambda}{}_{\alpha}\bar{g}^{\mu}{}_{\beta}T^{\alpha\beta}\iota_\omega\eta \\
&+ 2(-1)^n \int_{\Sigma(s_o)} \sigma_{\cdot}^{\kappa_1} \cdots \sigma_{\cdot}^{\kappa_n}\dot{z}^{\nu}\bar{g}^{(\lambda}{}_{\alpha,\nu}\bar{g}^{\mu)}{}_{\beta}T^{\alpha\beta}\iota_\omega\eta \\
&+ 2(-1)^n \int_{\Sigma(s_o)} \sigma_{\cdot}^{\kappa_1} \cdots \sigma_{\cdot}^{\kappa_n}\omega^{\gamma}\bar{g}^{(\lambda}{}_{\alpha,\gamma}\bar{g}^{\mu)}{}_{\beta}T^{\alpha\beta}\iota_\omega\eta \\
&+ (-1)^n \int_{\Sigma(s_o)} \sigma_{\cdot}^{\kappa_1} \cdots \sigma_{\cdot}^{\kappa_n}\bar{g}^{\lambda}{}_{\alpha}\bar{g}^{\mu}{}_{\beta}\omega^{\gamma}T^{\alpha\beta}{}_{,\gamma}\iota_\omega\eta
\end{aligned} \tag{16}$$

and so

$$\begin{aligned}
\dot{\mathfrak{t}}^{\kappa_1\cdots\kappa_n\lambda\mu}(s_o) &= n(-1)^n \int_{\Sigma(s_o)} \dot{z}^{\nu}\sigma_{\cdot}^{(\kappa_1}{}_{\nu}\sigma_{\cdot}^{\kappa_2} \cdots \sigma_{\cdot}^{\kappa_n)}\bar{g}^{\lambda}{}_{\alpha}\bar{g}^{\mu}{}_{\beta}T^{\alpha\beta}\iota_\omega\eta \\
&+ n(-1)^n \int_{\Sigma(s_o)} \omega^{\gamma}\sigma_{\cdot}^{(\kappa_1}{}_{\gamma}\sigma_{\cdot}^{\kappa_2} \cdots \sigma_{\cdot}^{\kappa_n)}\bar{g}^{\lambda}{}_{\alpha}\bar{g}^{\mu}{}_{\beta}T^{\alpha\beta}\iota_\omega\eta \\
&+ 2(-1)^n \int_{\Sigma(s_o)} \sigma_{\cdot}^{\kappa_1} \cdots \sigma_{\cdot}^{\kappa_n}\dot{z}^{\nu}\bar{g}^{(\lambda}{}_{\alpha.\nu}\bar{g}^{\mu)}{}_{\beta}T^{\alpha\beta}\iota_\omega\eta \\
&+ 2(-1)^n \int_{\Sigma(s_o)} \sigma_{\cdot}^{\kappa_1} \cdots \sigma_{\cdot}^{\kappa_n}\omega^{\gamma}\bar{g}^{(\gamma}{}_{\alpha.\gamma}\bar{g}^{\mu)}{}_{\beta}T^{\alpha\beta}\iota_\omega\eta
\end{aligned}$$

$$+ (-1)^n \int_{\Sigma(s_o)} \sigma_{.}^{\kappa_1} \cdots \sigma_{.}^{\kappa_n} \bar{g}^{\lambda}{}_{\alpha} \bar{g}^{\mu}{}_{\beta} T^{\alpha\beta}{}_{;\gamma} \omega^{\gamma} \iota_{\omega} \eta \tag{17}$$

Equations (4), (7), and (17) prove that the moments $\mathfrak{t}^{\cdots}$ have constant components with respect to a comoving orthonormal frame. ■

Finally, we study the evolution of the tensor fields $\mathfrak{p}^{\cdots}$ along L under L-rigidity conditions. We consider the vector field family along L

$$\bar{N}^{\kappa}(s, m) = \bar{g}^{\kappa}{}_{\alpha}(z(s), \gamma_m(s - s_m)) N^{\alpha}(\gamma_m(s - s_m)) \tag{18}$$

for $m \in \Sigma$. This family is defined analogously to the tensor field family $\bar{T}^{\lambda\mu}$ [equation (5)], and it is obtained by parallely transporting the normal vector field N^{α} from γ_m to L.

Applying the derivative operator $\overset{\mathrm{RM}}{D}_n$ to the last family at the point (s_m, m), we get

$$\overset{\mathrm{RM}}{D}_n \bar{N}^{\kappa} = (\bar{g}^{\kappa}{}_{\alpha.\lambda} \dot{z}^{\lambda} + \bar{g}^{\kappa}{}_{\alpha.\beta} \omega^{\beta} + M^{\kappa}{}_{\lambda} \bar{g}^{\lambda}{}_{\alpha}) N^{\alpha} + \bar{g}^{\kappa}{}_{\alpha} N^{\alpha}{}_{;\beta} \omega^{\beta} \tag{19}$$

where the bitensors are located at the point (z_m, m), the tensor fields along L at the point s_m, and the tensor fields at the point m.

Theorem 2. In L-rigid motion satisfying $\overset{\mathrm{RM}}{D}_n \bar{N}^{\kappa}(s_m, m) = 0$, for all $m \in \Sigma$, the moments $\mathfrak{p}^{\cdots}$ have constant components with respect to a comoving orthonormal frame.

Proof. For sufficiently small s and s_o arbitrary, taking into account that the flow φ_s of the vector field ω^{α} is a diffeomorphism that drags $\Sigma(s_o)$ into $\Sigma(s_o + s)$ and it preserves the orientation, we obtain

$$\mathfrak{p}^{\kappa_1 \cdots \kappa_n \lambda}(s_o + s) = (-1)^{n+1} \int_{\Sigma(s_o)} (\sigma_{.}^{\kappa_1} \cdots \sigma_{.}^{\kappa_n} \bar{g}^{\lambda}{}_{\alpha} T^{\alpha\beta} N_{\beta}) \circ \varphi_s \varphi_s^{*}(\iota_N \eta) \tag{20}$$

Taking the derivative at $s = 0$, taking into account equations (1) and (3), we get

$$\left.\frac{d\mathfrak{p}^{\kappa_1 \cdots \kappa_n \lambda}}{ds}\right|_{s_o} = n(-1)^{n+1} \int_{\Sigma(s_o)} \dot{z}^{\mu} \sigma_{.}^{(\kappa_1}{}_{,\mu} \sigma_{.}^{\kappa_2} \cdots \sigma_{.}^{\kappa_n)} \bar{g}^{\lambda}{}_{\alpha} T^{\alpha\beta} N_{\beta} \iota_N \eta$$

$$+ n(-1)^{n+1} \int_{\Sigma(s_o)} \omega^{\gamma} \sigma_{.}^{(\kappa_1}{}_{,\gamma} \sigma_{.}^{\kappa_2} \cdots \sigma_{.}^{\kappa_n)} \bar{g}^{\lambda}{}_{\alpha} T^{\alpha\beta} N_{\beta} \iota_N \eta$$

$$+ (-1)^{n+1} \int_{\Sigma(s_o)} \sigma_{.}^{\kappa_1} \cdots \sigma_{.}^{\kappa_n} \bar{g}^{\lambda}{}_{\alpha,\mu} \dot{z}^{\mu} T^{\alpha\beta} N_{\beta} \iota_N \eta$$

$$+ (-1)^{n+1} \int_{\Sigma(s_o)} \sigma_{.}^{\kappa_1} \cdots \sigma_{.}^{\kappa_n} \bar{g}^{\lambda}{}_{\alpha,\gamma} \omega^{\gamma} T^{\alpha\beta} N_{\beta} \iota_N \eta$$

$$+ (-1)^{n+1} \int_{\Sigma(s_o)} \sigma_{.}^{\kappa_1} \cdots \sigma_{.}^{\kappa_n} \bar{g}^{\lambda}{}_{\alpha} \omega^{\gamma} (T^{\alpha\beta} N_{\beta})_{,\gamma} \iota_N \eta \qquad (21)$$

and thus

$$\dot{p}^{\kappa_1 \cdots \kappa_n \lambda}(s_o) = n(-1)^{n+1} \int_{\Sigma(s_o)} \dot{z}^{\mu} \sigma_{.}^{(\kappa_1}{}_{\mu} \sigma_{.}^{\kappa_2} \cdots \sigma_{.}^{\kappa_n)} \bar{g}^{\lambda}{}_{\alpha} T^{\alpha\beta} N_{\beta} \iota_N \eta$$

$$+ n(-1)^{n+1} \int_{\Sigma(s_o)} \omega^{\gamma} \sigma_{.}^{(\kappa_1}{}_{\gamma} \sigma_{.}^{\kappa_2} \cdots \sigma_{.}^{\kappa_n)} \bar{g}^{\lambda}{}_{\alpha} T^{\alpha\beta} N_{\beta} \iota_N \eta$$

$$+ (-1)^{n+1} \int_{\Sigma(s_o)} \sigma_{.}^{\kappa_1} \cdots \sigma_{.}^{\kappa_n} \bar{g}^{\lambda}{}_{\alpha.\mu} \dot{z}^{\mu} T^{\alpha\beta} N_{\beta} \iota_N \eta$$

$$+ (-1)^{n+1} \int_{\Sigma(s_o)} \sigma_{.}^{\kappa_1} \cdots \sigma_{.}^{\kappa_n} \bar{g}^{\gamma}{}_{\alpha.\gamma} \omega^{\gamma} T^{\alpha\beta} N_{\beta} \iota_N \eta$$

$$+ (-1)^{n+1} \int_{\Sigma(s_o)} \sigma_{.}^{\kappa_1} \cdots \sigma_{.}^{\kappa_n} \bar{g}^{\lambda}{}_{\alpha} (T^{\alpha\beta} N_{\beta})_{;\gamma} \omega^{\gamma} \iota_N \eta \qquad (22)$$

Hence our theorem follows from equations (4), (7), (19), and (22). ■

From these two theorems, supposing that the vector field family $\bar{N}^{\kappa}$ along L satisfies rotating M-transport equations, we obtain that L-rigidity leads to the result that the multipole moments of the energy-momentum tensor, defined from the parallel propagator, have constant components with respect to a comoving orthonormal frame. In this sense, quasirigidity and L-rigidity are analogous concepts, so that quasirigidity imposes that the reduced multipole moments of the energy-momentum tensor have constant components with respect to a comoving orthonormal frame.

4. RELATION BETWEEN QUASIRIGIDITY AND *L*-RIGIDITY

In this section we study the relation between quasirigidity and L-rigidity in space-times of constant curvature (nonzero) and in space-times with small curvature (weak fields). We give the expression for the reduced multipole moments of the energy-momentum tensor in such space-times. We consider the covariant expansions of the second covariant derivatives of the world function and the Jacobi propagators.

4.1. *L*-Rigidity in Space-Times of Constant Nonzero Curvature

A space-time of constant curvature is characterized by a Riemann tensor with the components

$$R_{\alpha\beta\gamma\delta} = k(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{23}$$

where $k = R/12$.

In a space-time of constant curvature it is immediate to prove that

$$\sigma_{.\lambda\alpha} = -\bar{g}_{\lambda\alpha} - \frac{k}{6}(\bar{g}_{\lambda\alpha}\sigma_{.\rho}\sigma_{.}^{\rho} - \sigma_{.\lambda}\bar{g}_{\mu\alpha}\sigma_{.}^{\mu}) + O(s^3) \tag{24}$$

$$H_{\alpha\lambda} = \bar{g}_{\lambda\alpha} - \frac{k}{6}(\bar{g}_{\lambda\alpha}\sigma_{.\rho}\sigma_{.}^{\rho} - \sigma_{.\lambda}\bar{g}_{\mu\alpha}\sigma_{.}^{\mu}) + O(s^3) \tag{25}$$

$$K_{\alpha\lambda} = \bar{g}_{\lambda\alpha} - \frac{k}{2}(\bar{g}_{\lambda\alpha}\sigma_{.\rho}\sigma_{.}^{\rho} - \sigma_{.\lambda}\bar{g}_{\mu\alpha}\sigma_{.}^{\mu}) + O(s^3) \tag{26}$$

From these equations the bitensor $\underset{n}{\Theta}^{\kappa\lambda\mu\nu}$ [equation (9.5), (Dixon, 1974)] used to define the moments $J^{\cdots}$ is expressed as

$$\underset{n}{\Theta}^{\kappa\lambda\mu\nu} = g^{\kappa(\mu}g^{\nu)\lambda} - \frac{k}{3}\frac{n-1}{n+1}(g^{\kappa(\mu}\sigma_{.}^{\nu)}\sigma_{.}^{\lambda}$$
$$- 2g^{\kappa(\mu}g^{\nu)\lambda}\sigma_{.\rho}\sigma_{.}^{\rho} + \sigma_{.}^{\kappa}\sigma_{.}^{(\mu}g^{\nu)\lambda}) + O(s^3) \tag{27}$$

From equations (24)–(27) we get that the reduced multipole moments of the energy-momentum tensor, in a space-time of constant curvature, have the following components:

$$t^{\kappa_1\cdots\kappa_n\lambda\mu} = \mathrm{t}^{\kappa_1\cdots\kappa_n\lambda\mu} + \frac{k}{3}(\mathrm{t}^{\kappa_1\cdots\kappa_n}{}_{\rho}{}^{\rho\lambda\mu} - \mathrm{t}^{\kappa_1\cdots\kappa_n\rho(\lambda\mu)\rho}) + O(s^3) \tag{28}$$

$$p^{\kappa_1\cdots\kappa_n\lambda\mu\nu} = 2\mathrm{p}^{\kappa_1\cdots\kappa_n(\lambda}g^{\mu)\nu} - \frac{2k}{3}\frac{n-1}{n+1}\mathrm{p}^{\kappa_1\cdots\kappa_n\nu(\lambda\mu)}$$
$$+ \frac{k}{3}\frac{3n-5}{n+1}\mathrm{p}^{\kappa_1\cdots\kappa_n}{}_{\rho}{}^{\rho(\lambda}g^{\mu)\nu} - \frac{k}{3}\frac{n-3}{n+1}g^{\nu(\lambda}\mathrm{p}^{\nu)\kappa_1\cdots\kappa_n}{}_{\rho}{}^{\rho} + O(s^3) \tag{29}$$

We also obtain that the linear and angular momenta have the following components:

$$p^{\lambda} = \mathrm{p}^{\lambda} - kg_{\mu\nu}\mathrm{p}^{\mu[\nu\lambda]} + O(s^3) \tag{30}$$

$$S^{\lambda\mu} = -2\mathrm{p}^{[\lambda\mu]} + \frac{k}{3}g_{\nu\rho}\mathrm{p}^{\nu\rho[\lambda\mu]} + O(s^3) \tag{31}$$

On the other hand, the vector field family along L, $\overline{N}^\kappa$, has the expression

$$\overline{N}^\kappa(s, m) = n^\kappa(s) + O(\mathsf{s}^3) \tag{32}$$

Hence, or more exactly from equation (19), we have

$$\overset{\mathrm{RM}}{D}_n \overline{N}^\kappa \sim O(\mathsf{s}^2) \tag{33}$$

which indicates that, to the approximation order considered, the vector field family $\overline{N}^\kappa$ along L has constant components with respect to a comoving orthonormal frame.

Now we state the following theorem.

Theorem 3. In space-times of constant curvature the moments $\mathsf{p}^{\cdots}$ for $n \geq 1$ have constant components with respect to a comoving orthonormal frame under L-rigidity conditions to $O(\mathsf{s}^3)$. For $n = 0$ we get the same result, but to $O(\mathsf{s}^2)$.

Proof. The result follows from Theorem 2 and equations (29) and (33). ■

Because of this theorem, or more exactly using the same technique used to prove Theorems 1 and 2, and taking into account equations (30) and (31), we get the following consequence.

Corollary 4. In space-times of constant curvature the linear and angular momenta have constant components with respect to a comoving orthonormal frame under L-rigidity conditions to $O(\mathsf{s}^2)$ and $O(\mathsf{s}^3)$, respectively.

Considering the center-of-mass line L_o, leaving aside the unphysical case $S^2 = -12M^2/R$, Ehlers and Rudolph (1977) obtained

$$z^\lambda = n^\lambda \tag{34}$$

Hence we have the following result.

Theorem 5. In space-times of constant curvature L-rigidity leads to quasirigidity to $O(\mathsf{s}^3)$.

Proof. The proof is analogous to those of Theorems 1 and 2 by taking into account equations (28), (29), (33), and (34). ■

4.2. L-Rigidity in Weak Fields

We suppose that the curvature and its variation over space-time are small enough in a neighborhood of the body. Under this hypothesis, by using the covariant expansion of bitensors, it is proved that

$$\sigma_{.\lambda\alpha} = -\bar{g}_{\lambda\alpha} + O(\mathsf{s}^2) \tag{35}$$

$$H_{\alpha\lambda} = \bar{g}_{\lambda\alpha} + O(\mathsf{s}^2) \tag{36}$$

$$K_{\alpha\lambda} = \bar{g}_{\lambda\alpha} + O(\mathsf{s}^2) \tag{37}$$

From these equations the bitensor $\underset{n}{\Theta}^{\kappa\lambda\mu\nu}$ [equation (9.5), (Dixon, 1974)] used to define the moments $J^{\cdots}$ is expressed as

$$\underset{n}{\Theta}^{\kappa\lambda\mu\nu} = g^{\kappa(\mu}g^{\nu)\lambda} + O(\mathsf{s}^2) \tag{38}$$

From equations (35)–(38) we deduce that the reduced multipole moments of the energy-momentum tensor have the following components:

$$t^{\kappa_1\cdots\kappa_n\lambda\mu} = \mathsf{t}^{\kappa_1\cdots\kappa_n\lambda\mu} + O(\mathsf{s}^2) \tag{39}$$

$$p^{\kappa_1\cdots\kappa_n\lambda\mu\nu} = 2\mathsf{p}^{\kappa_1\cdots\kappa_n(\lambda}g^{\mu)\nu} + O(\mathsf{s}^2) \tag{40}$$

whereas the linear and angular momenta are

$$p^{\lambda} = \mathsf{p}^{\lambda} + O(\mathsf{s}^2) \tag{41}$$

$$S^{\lambda\mu} = -2\mathsf{p}^{[\lambda\mu]} + O(\mathsf{s}^2) \tag{42}$$

On the other hand, the vector field family along L, $\bar{N}^{\lambda}$, has the following components:

$$\bar{N}^{\lambda} = n^{\lambda} + O(\mathsf{s}^2) \tag{43}$$

Then, from this equation or taking into account (19), we get

$$\overset{\text{RM}}{D}_n \bar{N}^{\kappa} \sim O(\mathsf{s}) \tag{44}$$

Hence we obtain the following result.

Theorem 6. In space-times with small curvature the moments $\mathsf{p}^{\cdots}$ for $n \geq 1$ have constant components with respect to a comoving orthonormal frame under L-rigidity conditions to $O(\mathsf{s}^2)$. For $n = 0$ we get the same result, but to $O(\mathsf{s})$.

Proof. The proof is obtained from Theorem 2 and equations (29) and (44). ∎

Because of this theorem and taking into account equations (41) and (42), we get the following consequence.

Corollary 7. In space-times with small curvature the linear and angular momenta have constant components with respect to a comoving orthonormal frame under L-rigidity conditions to $O(\mathsf{s})$ and $O(\mathsf{s}^2)$, respectively.

We now consider the center-of-mass line L_o. Then from Theorem 6 we get

$$\dot{M} \sim O(\mathsf{s}) \tag{45}$$

that is, the mass is a constant of motion, under L-rigidity conditions, to this approximation order.

Moreover, taking into account equation (2.17) (Ehlers and Rudolph, 1977), we get

$$\dot{z}^\lambda = n^\lambda + O(\mathsf{s}^2) \tag{46}$$

To conclude, we get the following theorem.

Theorem 8. L-rigidity under weak fields conditions leads to quasirigidity to $O(\mathsf{s}^2)$.

Proof. The proof is analogous to those of Theorems 1 and 2 by taking into account equations (39), (40), (44), and (46). ■

5. DISCUSSION

In a previous paper (Barreda and Olivert, 1996) we introduced L-rigidity, which, like quasirigidity, is a specialization of pseudorigidity, and hence we are interested in relating these two concepts.

As indicated in Section 3, L-rigidity is analogous to quasirigidity under a weak condition in the sense that both preserve a set of multipole moments of the energy-momentum tensor: quasirigidity the reduced multipole moments and L-rigidity the multipole moments defined from the parallel propagator.

In space-times of constant curvature we obtain that L-rigidity leads to an increase of two orders of magnitude in the rate of change with respect to a comoving orthonormal frame of the linear momentum and an increase of three orders of magnitude in the rate of change with respect to a comoving orthonormal frame of the angular momentum. A more important consequence is that L-rigidity leads to quasirigidity to $O(\mathsf{s}^3)$.

In weak fields we also obtain an increase of one order of magnitude in the rate of change with respect to a comoving orthonormal frame of the linear momentum and an increase of two orders of magnitude in the rate of change with respect to a comoving orthonormal frame of the angular momentum. Moreover, L-rigidity leads to quasirigidity to $O(\mathsf{s}^2)$.

These results are a consequence of the fact that the reduced multipole moments of the energy-momentum tensor and the linear and angular momenta are expressed in such space-times from the multipole moments of the energy-momentum tensor defined from parallel propagator which, under *L*-rigidity conditions, have constant components with respect to a comoving orthonormal frame to the approximation order considered.

We have assumed that the rotating M-transport operator $\overset{\mathrm{RM}}{D}_n$ decreases the original approximation orders in one unit when acting over moments of $O(\mathsf{s})$ or higher, while it preserves the original approximation orders when acting over moments of order 0 (linear momentum).

The relation between quasirigidity and *L*-rigidity cannot be studied in the pole-dipole approximation (Dixon, 1964) since at this level of approximation the reduced multipole moments of the energy-momentum tensor vanish. We consider covariant expansion of bitensors about a point, and we only neglect higher order terms in obtaining expansions with respect to the parameter s (geodetic interval) of the linear and angular momenta and of the reduced multipole moments of the energy-momentum tensor and its rate of change with respect to a comoving orthonormal frame.

Equations of motion of an *L*-rigid body and degrees of freedom of *L*-rigid motion will be treated in a subsequent paper.

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